Solution to HW9

Leon Li

ylli@math.cuhk.edu.hk

MATH 2020 B   HW 9

Due Date: Apr 24, 2020 (12:00 noon)

Thomas' Calculus (12th Ed.)

§16.5: 3, 7, 18, 21, 29, 34, 56(b) (using 56(a))

§16.6: 14, 20, 22.

§16.5

Finding Parametrizations

In Exercises 1–16, find a parametrization of the surface. (There are many correct ways to do these, so your answers may not be the same as those in the back of the book.)

3. Cone frustum The first-octant portion of the cone $z = \sqrt{x^2 + y^2}/2$ between the planes $z = 0$ and $z = 3$

7. Spherical band The portion of the sphere $x^2 + y^2 + z^2 = 3$ between the planes $z = \sqrt{3}/2$ and $z = -\sqrt{3}/2$

Sol) (3) Using cylindrical coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, where $r \geq 0$ and $0 \leq \theta < 2\pi$. Then

- $z = \frac{1}{2}\sqrt{x^2+y^2} \iff z = \frac{r}{2}$. Hence $0 \leq z \leq 3 \iff 0 \leq r \leq 6$.

- (x, y, z) lies in first octant $\iff 0 \leq \theta \leq \frac{\pi}{2}$

$\therefore$ A parametrization is given by $\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + \frac{r}{2}\vec{k}$, where

$0 \leq r \leq 6$ and $0 \leq \theta \leq \frac{\pi}{2}$.

(7) Using spherical coordinates $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$, where $\begin{cases} \rho \geq 0 \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta < 2\pi \end{cases}$. Then

- $x^2+y^2+z^2 = 3 \iff \rho = \sqrt{3}$

- $-\frac{\sqrt{3}}{2} \leq z \leq \frac{\sqrt{3}}{2} \iff -\frac{\sqrt{3}}{2} \leq \rho\cos\phi \leq \frac{\sqrt{3}}{2} \iff -\frac{1}{2} \leq \cos\phi \leq \frac{1}{2} \iff \frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$.

$\therefore$ A parametrization is given by $\vec{r}(\phi,\theta) = \sqrt{3}\sin\phi\cos\theta\,\vec{i} + \sqrt{3}\sin\phi\sin\theta\,\vec{j} + \sqrt{3}\cos\phi\,\vec{k}$,

where $\frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$ and $0 \leq \theta < 2\pi$.

Surface Area of Parametrized Surfaces

In Exercises 17–26, use a parametrization to express the area of the surface as a double integral. Then evaluate the integral. (There are many correct ways to set up the integrals, so your integrals may not be the same as those in the back of the book. They should have the same values, however.)

18. Plane inside cylinder The portion of the plane $z = -x$ inside the cylinder $x^2 + y^2 = 4$

Sol) Using cylindrical coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, where $r \geq 0$ and $0 \leq \theta < 2\pi$. Then

- $x^2+y^2 \leq 4 \iff 0 \leq r \leq 2$

- $z = -x \iff z = -r\cos\theta$

$\therefore$ A parametrization is given by $\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} - r\cos\theta\,\vec{k}$, where

$0 \leq r \leq 2$ and $0 \leq \theta < 2\pi$.

$\vec{r}_r(r,\theta) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} - \cos\theta\,\vec{k}$; $\vec{r}_\theta(r,\theta) = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j} + r\sin\theta\,\vec{k}$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -\cos\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta \end{vmatrix}$$

$$= (r\sin^2\theta + r\cos^2\theta)\vec{i} - (r\sin\theta\cos\theta - r\sin\theta\cos\theta)\vec{j} + (r\cos^2\theta + r\sin^2\theta)\vec{k} = r\vec{i} + r\vec{k}.$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2+r^2} = \sqrt{2}\,r.$$

$$\therefore \text{Area} = \int_0^{2\pi}\int_0^2 (\sqrt{2}r)\,dr\,d\theta = 2\pi \cdot \left[\frac{\sqrt{2}}{2}r^2\right]_0^2 = 4\pi\sqrt{2}$$

21. Circular cylinder band The portion of the cylinder $x^2 + y^2 = 1$ between the planes $z = 1$ and $z = 4$

Sol) Using cylindrical coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, where $r \geq 0$ and $0 \leq \theta < 2\pi$. Then

- $x^2+y^2=1 \iff r=1$
- $1 \leq z \leq 4 \iff 1 \leq z \leq 4$

$\therefore$ A parametrization is given by $\vec{r}(\theta, z) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + z\vec{k}$, where

$1 \leq z \leq 4$ and $0 \leq \theta < 2\pi$.

$\vec{r}_\theta(\theta,z) = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$; $\vec{r}_z(\theta,z) = \vec{k}$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}.$$

$$|\vec{r}_\theta \times \vec{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

$$\therefore \text{Area} = \int_1^4 \int_0^{2\pi} d\theta\, dz = 3 \cdot 2\pi = 6\pi$$

Planes Tangent to Parametrized Surfaces

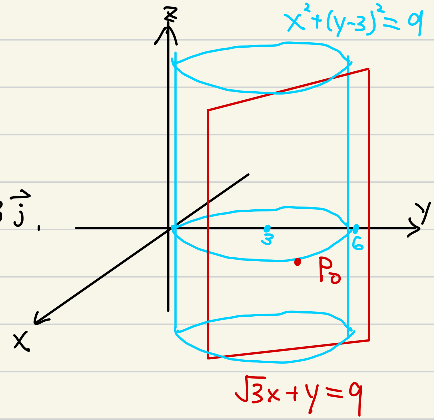
The tangent plane at a point $P_0(f(u_0, v_0), g(u_0, v_0), h(u_0, v_0))$ on a parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is the plane through P_0 normal to the vector $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$, the cross product of the tangent vectors $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ at P_0. In Exercises 27–30, find an equation for the plane tangent to the surface at P_0. Then find a Cartesian equation for the surface and sketch the surface and tangent plane together.

29. Circular cylinder The circular cylinder $\mathbf{r}(\theta, z) = (3\sin 2\theta)\mathbf{i} + (6\sin^2\theta)\mathbf{j} + z\mathbf{k}$, $0 \le \theta \le \pi$, at the point $P_0\left(3\sqrt{3}/2, 9/2, 0\right)$ corresponding to $(\theta, z) = (\pi/3, 0)$ (See Example 3.)

Sol) $\vec{r}(\theta, z) = (3\sin 2\theta)\vec{i} + (6\sin^2\theta)\vec{j} + z\vec{k}$.

$\vec{r}_\theta(\theta, z) = (6\cos 2\theta)\vec{i} + (12\sin\theta\cos\theta)\vec{j}$; $\vec{r}_\theta(\frac{\pi}{3}, 0) = -3\vec{i} + 3\sqrt{3}\vec{j}$;

$\vec{r}_z(\theta, z) = \vec{k}$; $\vec{r}_z(\frac{\pi}{3}, 0) = \vec{k}$.



$$\vec{r}_\theta(\tfrac{\pi}{3}, 0) \times \vec{r}_z(\tfrac{\pi}{3}, 0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 3\sqrt{3} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\sqrt{3}\vec{i} + 3\vec{j}.$$

$\therefore$ Equation of the Tangent plane at P_0 is

$(3\sqrt{3}\vec{i} + 3\vec{j}) \cdot ((x - \frac{3\sqrt{3}}{2})\vec{i} + (y - \frac{9}{2})\vec{j} + \vec{k}) = 0$, i.e. $\sqrt{3}x + y = 9$ //

Finding Cartesian equation of the surface: let $x = 3\sin 2\theta = 6\sin\theta\cos\theta$; $y = 6\sin^2\theta$.

Then $x^2 + y^2 = 36(\sin^2\theta\cos^2\theta + \sin^4\theta) = 36\sin^2\theta = 6y$. $\therefore x^2 + (y-3)^2 = 9$ //

More Parametrizations of Surfaces

34. Hyperboloid of one sheet

a. Find a parametrization for the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ in terms of the angle θ associated with the circle $x^2 + y^2 = r^2$ and the hyperbolic parameter u associated with the hyperbolic function $r^2 - z^2 = 1$. (*Hint*: $\cosh^2 u - \sinh^2 u = 1$.)

b. Generalize the result in part (a) to the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$.

Sol) (a) Let $\begin{cases} x = r\cos\theta = \cosh u \cos\theta \\ y = r\sin\theta = \cosh u \sin\theta \\ z = \sinh u \end{cases}$, where $0 \leq \theta < 2\pi$ and $u \in \mathbb{R}$.

Then $x^2+y^2 = r^2 = \cosh^2 u$; $x^2+y^2-z^2 = \cosh^2 u - \sinh^2 u = 1$.

$\therefore$ A parametrization is given by $\vec{r}(\theta, u) = \cosh u \cos\theta \, \vec{i} + \cosh u \sin\theta \, \vec{j} + \sinh u \, \vec{k}$,

where $0 \leq \theta < 2\pi$ and $u \in \mathbb{R}$.

(b) Let $\begin{cases} x = ax' \\ y = by' \\ z = cz' \end{cases}$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \Leftrightarrow x'^2 + y'^2 - z'^2 = 1$

By (a), $x'^2 + y'^2 - z'^2 = 1$ has a parametrization given by

$\vec{r}'(\theta, u) = \cosh u \cos\theta \, \vec{i} + \cosh u \sin\theta \, \vec{j} + \sinh u \, \vec{k}$, where $0 \leq \theta < 2\pi$ and $u \in \mathbb{R}$.

$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ has a parametrization given by

$\vec{r}(\theta, u) = a\cosh u \cos\theta \, \vec{i} + b\cosh u \sin\theta \, \vec{j} + c\sinh u \, \vec{k}$, where $0 \leq \theta < 2\pi$ and $u \in \mathbb{R}$.

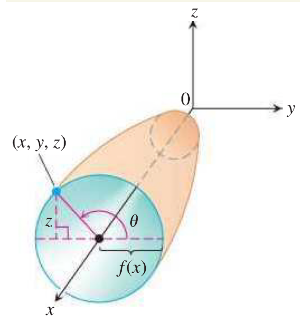
Surface Area for Implicit and Explicit Forms

56. Let S be the surface obtained by rotating the smooth curve $y = f(x)$, $a \le x \le b$, about the x-axis, where $f(x) \ge 0$.

a. Show that the vector function

$$\mathbf{r}(x, \theta) = x\mathbf{i} + f(x)\cos\theta\,\mathbf{j} + f(x)\sin\theta\,\mathbf{k}$$

is a parametrization of S, where θ is the angle of rotation around the x-axis (see the accompanying figure).



b. Use Equation (4) to show that the surface area of this surface of revolution is given by

$$A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

Sol) (a) Assumed to be true in HW9.

(b) $\vec{r}(x,\theta) = x\vec{i} + f(x)\cos\theta\,\vec{j} + f(x)\sin\theta\,\vec{k}$, where $a \le x \le b$ and $0 \le \theta < 2\pi$

$\vec{r}_x(x,\theta) = \vec{i} + f'(x)\cos\theta\,\vec{j} + f'(x)\sin\theta\,\vec{k}$. $\vec{r}_\theta(x,\theta) = -f(x)\sin\theta\,\vec{j} + f(x)\cos\theta\,\vec{k}$.

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix}$$

$$= (f'(x)f(x)\cos^2\theta + f'(x)f(x)\cos^2\theta)\vec{i} - (f(x)\cos\theta)\vec{j} + (-f(x)\sin\theta)\vec{k}$$

$$= (f'(x)f(x))\vec{i} - (f(x)\cos\theta)\vec{j} + (-f(x)\sin\theta)\vec{k}$$

$$|\vec{r}_x \times \vec{r}_\theta| = \sqrt{(f'(x)f(x))^2 + (f(x)\cos\theta)^2 + (f(x)\sin\theta)^2} = f(x)\sqrt{1 + (f'(x))^2}$$

$$\therefore A = \int_0^{2\pi}\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx\,d\theta = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx \;//$$

§16.6

Surface Integrals

14. Integrate $G(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0$, $x = 1$, and $z = 0$.

Sol) Let $f(x,y,z) = y^2+4z$; $\nabla f(x,y,z) = 2y\vec{j} + 4\vec{k}$;

$\nabla f \cdot \vec{k} = 4$; $|\nabla f \cdot \vec{k}| = 4$; $|\nabla f(x,y,z)| = \sqrt{(2y)^2 + 4^2} = 2\sqrt{y^2+4}$

$\therefore d\sigma = \frac{2\sqrt{y^2+4}}{4}\,dx\,dy = \frac{1}{2}\sqrt{y^2+4}\,dx\,dy.$

Also, when $z=0$, $y^2 = 16 \Leftrightarrow y = \pm 4$.

$\therefore$ The surface integral $= \int_{-4}^{4}\int_0^1 \left(x\sqrt{y^2+4}\right)\left(\frac{1}{2}\sqrt{y^2+4}\right)dx\,dy$

$= \frac{1}{2}\left(\int_{-4}^{4}(y^2+4)\,dy\right)\left(\int_0^1 x\,dx\right) = \frac{1}{2}\left[\frac{y^3}{3}+4y\right]_{-4}^{4}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}\cdot\frac{224}{3}\cdot\frac{1}{2} = \frac{56}{3}$ //

Finding Flux Across a Surface

In Exercises 19–28, use a parametrization to find the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ across the surface in the given direction.

20. Parabolic cylinder $\mathbf{F} = x^2\mathbf{j} - xz\mathbf{k}$ outward (normal away from the yz-plane) through the surface cut from the parabolic cylinder $y = x^2$, $-1 \le x \le 1$, by the planes $z = 0$ and $z = 2$

22. Sphere $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$ in the direction away from the origin

Sol) (20) A parametrization is given by $\vec{r}(x,z) = x\vec{i} + x^2\vec{j} + z\vec{k}$,

where $-1 \le x \le 1$ and $0 \le z \le 2$; $\vec{r}_x(x,z) = \vec{i} + 2x\vec{j}$; $\vec{r}_z(x,z) = \vec{k}$.

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\vec{i} - \vec{j} \ ; \ \vec{F}(\vec{r}(x,z)) = x^2\vec{j} - xz\vec{k}.$$

$$\vec{F}(\vec{r}(x,z)) \cdot (\vec{r}_x \times \vec{r}_z) = -x^2.$$

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, d\sigma = \int_{-1}^{1}\int_0^2 (-x^2)\,dz\,dx = 2 \cdot \left[-\frac{x^3}{3}\right]_{-1}^{1} = -\frac{4}{3} \,//$$

(22) A parametrization is given by $\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\,\vec{i} + a\sin\phi\sin\theta\,\vec{j} + a\cos\phi\,\vec{k}$,

where $0 \le \phi \le \pi$ and $0 \le \theta < 2\pi$.

$\vec{r}_\phi(\phi,\theta) = a\cos\phi\cos\theta\,\vec{i} + a\cos\phi\sin\theta\,\vec{j} - a\sin\phi\,\vec{k}$; $\vec{r}_\theta(\phi,\theta) = -a\sin\phi\sin\theta\,\vec{i} + a\sin\phi\cos\theta\,\vec{j}$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix} = a^2\sin\phi \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin\phi\left(\sin\phi\cos\theta\,\vec{i} - \sin\phi\sin\theta\,\vec{j} + (\cos\phi\cos^2\theta + \cos\phi\sin^2\theta)\vec{k}\right)$$

$$= a^2\sin\phi\left(\sin\phi\cos\theta\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \cos\phi\,\vec{k}\right).$$

$$\vec{F}(\vec{r}(\phi,\theta)) = a\sin\phi\cos\theta\,\vec{i} + a\sin\phi\sin\theta\,\vec{j} + a\cos\phi\,\vec{k}.$$

$$\vec{F}(\vec{r}(\phi,\theta))\cdot(\vec{r}_\phi \times \vec{r}_\theta) = a^3\sin\phi(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi) = a^3\sin\phi.$$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \int_0^{2\pi}\int_0^{\pi} a^3\sin\phi\,d\phi\,d\theta = 2\pi a^3\cdot[-\cos\phi]_0^{\pi} = 4\pi a^3 /\!/$$